

Mixed, Multi-color, and Bipartite Ramsey Numbers Involving Trees of Small Diameter

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Abstract In this paper we study Ramsey numbers for trees of diameter 3 (bistars) vs., respectively, trees of diameter 2 (stars), complete graphs, and many complete graphs. In the case of bistars vs. many complete graphs, we determine this number exactly as a function of the Ramsey number for the complete graphs. We also determine the order of growth of the bipartite k -color Ramsey number for a bistar.

Keywords Ramsey numbers of trees · bipartite Ramsey numbers · stars · bistars

1 Introduction

1.1 Background

In this paper we investigate Ramsey numbers, both classical and bipartite, for trees vs. other graphs. Trees have been studied less than other graphs, although there have been a number of papers in the last few years. Some general results applying to all trees are known, such as the following result of Gyárfás and Tuza [4].

Theorem 1. *Let T_n be a tree with n edges. Then $R_k(T_n) \leq (n - 1)(k + \sqrt{k(k - 1)}) + 2$.*

More recently, various researchers have studied particular trees of small diameter. Burr and Roberts [3] completely determine the Ramsey number $R(S_{n_1}, \dots, S_{n_i})$ for any number of *stars*, i.e., trees of diameter 2. Boza et. al. [2] determine $R(S_{n_1}, \dots, S_{n_i}, K_{m_1}, \dots, K_{m_j})$ exactly as a function of $R(K_{m_1}, \dots, K_{m_j})$.

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Bahls and Spencer [1] study $R(C, C)$, where C is a caterpillar, i.e., a tree whose non-leaf vertices form a path. They prove a general lower bound, and prove exact results in several cases, including “regular” caterpillars, in which all non-leaf vertices have the same degree.

We will study bistars (i.e. trees of diameter 3) vs. stars and bistars vs. complete graphs in Section 2, bistars vs. many complete graphs in Section 3, and bistars vs. bistars in bipartite graphs in Section 4.

1.2 Notation

For graphs G_1, \dots, G_n , let $R(G_1, \dots, G_n)$ denote the least integer N such that any edge-coloring of K_N in n colors must contain, for some $1 \leq i \leq n$, a monochromatic G_i in the i^{th} color. Let S_n denote the $(n+1)$ -vertex graph consisting of a vertex v of degree n and n vertices of degree 1 (a star). Let $B_{k,m}$ denote the $(k+m)$ -vertex graph with a vertex v of degree k , a vertex w incident to v of degree m , and $k+m-2$ vertices of degree 1 (a *bistar*). We will call the edge vw the *spine* of $B_{k,m}$. (Note that some authors refer to the set of vertices $\{v, w\}$ as the spine.) We will depict the spine of a bistar with a double-struck edge; see Figure 1.

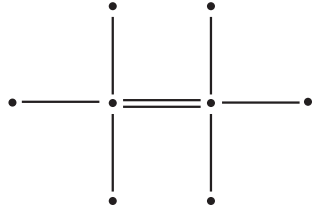


Fig. 1 A bistar, with spine indicated

For a graph G whose edges are colored red and blue, and for vertices v and w , if v and w are incident by a red edge, we will say (for the sake of brevity) that w is a “red neighbor” of v . Let $\deg_{\text{red}}(v)$ denote the number of red neighbors of v , and let

$$\Delta_{\text{red}}(G) = \max\{\deg_{\text{red}}(v) : v \in G\}$$

and

$$\delta_{\text{red}}(G) = \min\{\deg_{\text{red}}(v) : v \in G\}.$$

In Section 2 we will make use of *cyclic colorings*. Let K_N have vertex set $\{0, 1, 2, \dots, N-1\}$, and let $R \subseteq \mathbb{Z}_N \setminus 0$ such that $R = -R$, i.e., R is closed under additive inverse. Define a coloring of K_N by

uv is colored *red* if $u - v \in R$ and *blue* otherwise.

Cyclic colorings are computationally nice. For instance, it is not hard to show that if $R \subseteq R + R$, then any two vertices v and w incident by a red edge must share a red neighbor. We will need this fact in the proof of Theorem 3.

2 Mixed 2-Color Ramsey Numbers

First we consider bistars vs. stars. We have the following easy upper bound.

Theorem 2. $R(B_{k,m}, S_n) \leq k + m + n - 1$.

Proof. Let $N = k + m + n - 1$, and let the edges of K_N be colored in red and blue. Suppose this coloring contains no blue S_n . Then every red edge is the spine of a red $B_{k,m}$, as follows.

If there is no blue S_n , then $\Delta_{\text{blue}} \leq n - 1$, and hence $\delta_{\text{red}} \geq (N - 1) - (n - 1) = k + m - 1$. Let the edge uv be colored red. Then both u and v have $(k - 1) + (m - 1)$ red neighbors besides each other. Even if these sets of neighbors coincide, we may select $k - 1$ leaves for u and $m - 1$ leaves for v , giving a red $B_{k,m}$. \square

The following lower bound uses some cyclic colorings.

Theorem 3. $R(B_{k,m}, S_n) > \lfloor \frac{k+m}{2} \rfloor + n$ for $k, m \geq 4$.

Proof. Let $k + m$ be odd. Let $N = \lfloor \frac{k+m}{2} \rfloor + n$. Let G be any $(n - 1)$ -regular graph on N vertices. Consider the edges of G to be the *blue* edges, and replace all non-edges of G with red edges, so that the resulting K_N is $\lfloor \frac{k+m}{2} \rfloor$ -regular for red. Clearly, this coloring admits no blue S_n . Consider the red edge set. If an edge uv is colored red, then u and v combined have at most $k + m - 3$ red neighbors besides each other, which is not enough to supply the needed $k - 1$ red leaves for u and the $m - 1$ red leaves for v .

Now let $k + m$ be even, and $N = \frac{k+m}{2} + n$. We seek a subset $R \subseteq \mathbb{Z}_N$ that is symmetric ($R = -R$) and of size $\frac{k+m}{2}$ satisfying $R \subseteq R + R$. Thus each vertex will have red degree $\frac{k+m}{2}$, but any red edge uv cannot be the spine of a red $B_{k,m}$, since u and v will have a common neighbor. There are two cases:

Case (i.): $\frac{k+m}{2}$ is even. Let $R' = \{2\} \cup \{2\ell + 1 : 1 \leq \ell \leq \frac{k+m-4}{4}\}$, and let $R := R' \cup -R'$. It is easy to check that $R \subseteq R + R$. Setting $B = \mathbb{Z}_N \setminus \{R \cup 0\}$, we have $|B| = n - 1$, and so the cyclic coloring of K_N induced by R and B has no red $B_{k,m}$ and no blue S_n .

Case (ii.): $\frac{k+m}{2}$ is odd. Let $R' = \{2\} \cup \{2\ell + 1 : 1 \leq \ell \leq \frac{k+m-6}{4}\}$, and set $R := R' \cup \{\frac{k+m}{2}\} \cup -R'$. Again, set $B = \mathbb{Z}_N \setminus \{R \cup 0\}$, and the cyclic coloring of K_N induced by R and B has the desired properties. \square

Corollary 4. $R(B_{n,n}, S_n) > 2n$ for $n \geq 4$.

We conjecture that the lower bound in Corollary 4 is tight; that is, that $R(B_{n,n}, S_n) = 2n + 1$ for $n \geq 4$. We show that this result obtains for $n = 4$ (but not for $n = 3$).

Theorem 5. $R(B_{3,3}, S_3) = 6$.

Proof. A lower bound is supplied by the classic critical coloring of K_5 for $R(3,3)$. See Figure 2.

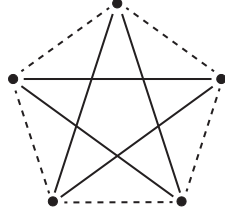


Fig. 2 Critical coloring of K_5

For the upper bound, suppose there exists a 2-coloring of K_6 with no blue S_3 . Then $\delta_{\text{red}} \geq 3$. So consider the red subgraph G . If G has a vertex of degree 5, the existence of a $B_{3,3}$ is immediate. If G has a vertex of degree 4, then it must have 2 such vertices u and v . If $u \sim v$, then G must look like Figure 3. One may use any edge incident to u or v as a spine.

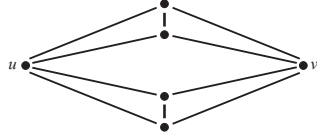


Fig. 3 Configuration of the red subgraph G .

If $u \sim v$, and u and v do not share all three remaining neighbors, then the existence of a $B_{3,3}$ is immediate. So suppose u and v have neighbors x, y , and z . The only way for G to have degree sequence $(4, 4, 3, 3, 3, 3)$ is for the remaining vertex w to be adjacent to x, y , and z . Then we have a $B_{3,3}$ as indicated in Figure 4. Finally, suppose G is 3-regular. If there is no $B_{3,3}$, then any adjacent vertices share a neighbor. It is not hard to see that adjacent vertices cannot share *two* neighbors in a 3-regular graph on 6 vertices. If any two adjacent vertices share exactly one neighbor, then G can be partitioned into edge-disjoint triangles. But any vertex in such a graph must have even degree, since its degree will be twice the number of triangles in which it participates. This is a contradiction, so there must be some vertices u and v that have no common neighbor. But u and v each have degree three, immediately yielding a $B_{3,3}$. \square

Theorem 6. $R(B_{4,4}, S_4) = 9$.

Proof. The lower bound is given by Theorem 3. For the upper bound, suppose a 2-coloring of K_9 contains no blue S_4 . Then $\delta_{\text{red}} \geq 5$. Let G be the red

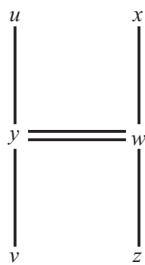


Fig. 4 A red $B_{3,3}$

subgraph. Since G has odd order, there must be at least one vertex v of degree ≥ 6 . Suppose $v \sim w$. It is easy to see that v and w must have at least two neighbors in common; call them y and z . Now v is adjacent to 3 other vertices; call them x_1, x_2 , and x_3 . There are two remaining vertices x_4 and x_5 . If w is adjacent to either of them, we are done. So suppose w is adjacent to x_1 and x_2 . If either x_4 or x_5 is adjacent to v , we are done, so suppose neither x_4 nor x_5 is adjacent to v or to w . Then x_4 (in order to have degree ≥ 5) must be adjacent to y_1 or to y_2 . Suppose it's y_1 . There are two cases:

1. $y_1 \sim x_5$. Then we have a red $B_{4,4}$ as indicated in Figure 5.

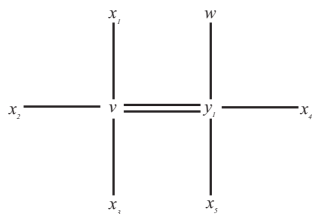


Fig. 5 A red $B_{4,4}$

2. $y_1 \approx x_5$. Then, since $\deg(y_1) \geq 5$, $y_1 \sim x_i$ for some $i \in \{1, 2, 3\}$. Then we have a red $B_{4,4}$ as indicated in Figure 6, where $|\{i, k, \ell\}| = 3$.

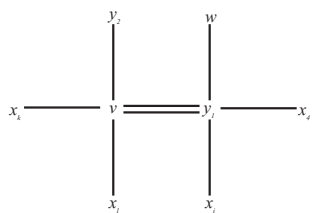


Fig. 6 A red $B_{4,4}$

□

Consideration of $R(B_{5,5}, S_5)$ leads into rather unpleasant case analysis when trying to reduce the upper bound from that given by Theorem 2.

Now we consider bistars vs. complete graphs.

Theorem 7. $R(B_{k,m}, K_3) = 2(k + m - 1) + 1$.

Proof. For the lower bound, let V_1 and V_2 be two red cliques, each of size $k + m - 1$, and let every edge between V_1 and V_2 be colored blue.

For the upper bound, let $N = 2(k + m - 1) + 1$, and give K_N an edge-coloring in red and blue. Suppose there is a vertex v with blue degree at least $k + m$. If any edge in $N_{\text{blue}}(v)$ is blue, we have a blue triangle. If not, then $N_{\text{blue}}(v)$ is a red clique of size at least $k + m$, so it contains a red $B_{k,m}$.

So then suppose that $\Delta_{\text{blue}} < k + m$. It follows that $\delta_{\text{red}} \geq k + m - 1$. Then every red edge is the spine of some red $B_{k,m}$. To see this, let uv be colored red. Both u and v each have at least $k + m - 2$ other red neighbors. Even if these red neighborhoods coincide, there are still $k - 1$ red leaves for u and $m - 1$ red leaves for v . □

Now we extend to arbitrary K_n .

Theorem 8. $R(B_{k,m}, K_n) = (k + m - 1)(n - 1) + 1$.

Proof. We proceed by induction on n . Theorem 7 provides the base case $n = 3$.

So assume $n > 3$, and let $R(B_{k,m}, K_{n-1}) \leq (k + m - 1)(n - 2) + 1$. Let $N = (k + m - 1)(n - 1) + 1$, and consider any edge-coloring of K_n in red and blue. If $\delta_{\text{red}} \geq k + m - 1$, then every red edge is the spine of a red $B_{k,m}$, so suppose $\delta_{\text{red}} \leq k + m - 2$. Then there is a vertex v with blue degree at least $(k + m - 1)(n - 2) + 1$. By the induction hypothesis, the subgraph induced by $N_{\text{blue}}(v)$ contains either a red $B_{k,m}$ or a blue K_{n-1} . In the latter case, the blue K_{n-1} along with v forms a blue K_n .

For the lower bound, let V_1, \dots, V_{n-1} be vertex-disjoint red cliques, each of size $k + m - 1$. Color all edges among the V_i 's blue. Clearly there are no red $B_{k,m}$'s. Since the blue subgraph forms a Turan graph, there are no blue K_n 's. □

3 Mixed Multi-color Ramsey Numbers

In [2], the authors determine $R(S_{k_1}, \dots, S_{k_i}, k_{n_i}, \dots, K_{n_\ell})$ exactly as a function of $R(K_{n_1}, \dots, K_{n_\ell})$. In [6], Omidi and Raeisi give a shorter proof of this result via the following lemma, whose proof is straight from The Book.

Lemma 9. *Let G_1, \dots, G_m be connected graphs, let $r = R(G_1, \dots, G_m)$ and $r' = R(K_{n_1}, \dots, K_{n_\ell})$. If $n \geq 2$ and $R(G_1, \dots, G_m, K_n) = (r - 1)(n - 1) + 1$, then $R(G_1, \dots, G_m, K_{n_1}, \dots, K_{n_\ell}) = (r - 1)(r' - 1) + 1$.*

Proof. Let $R = R(G_1, \dots, G_m, K_{n_1}, \dots, K_{n_\ell})$. For the lower bound, give $K_{r'-1}$ an edge-coloring in ℓ colors $\beta_1, \dots, \beta_\ell$ that has no copy of K_{n_i} in color β_i . Replace each vertex of $K_{r'-1}$ by a complete graph of order $r-1$ whose edges are colored by colors $\alpha_1, \dots, \alpha_m$ so that no copy of G_i appears in color α_i . Each edge in the original graph $K_{r'-1}$ expands to a copy of $K_{r-1, r-1}$, with each edge the same color as the original edge. This shows that $R > (r-1)(r'-1)$.

For the upper bound, let $N = (r-1)(r'-1) + 1$, and color the edges of K_N in colors $\alpha_1, \dots, \alpha_m, \beta_1, \dots, \beta_\ell$. Recolor the edges colored $\beta_1, \dots, \beta_\ell$ with a new color α . Since $R(G_1, \dots, G_m, K_{r'}) = (r'-1)(r-1) + 1 = N$, K_N contains a copy of G_i in color α_i or a copy of $K_{r'}$ in color α . In the former case we are done, so assume the latter obtains. Then consider the clique $K_{r'}$ which is colored α . Return to the original coloring in colors $\beta_1, \dots, \beta_\ell$. Since $R(K_{n_1}, \dots, K_{n_\ell}) = r'$, some color class β_i contains a copy of K_{n_i} . This concludes the proof. \square

We will now make use of Lemma 9 to determine $R(B_{k,m}, K_{n_1}, \dots, K_{n_\ell})$ as a function of $R(K_{n_1}, \dots, K_{n_\ell})$.

Theorem 10. $R(B_{k,m}, K_{n_1}, \dots, K_{n_\ell}) = (k+m-1)[R(K_{n_1}, \dots, K_{n_\ell}) - 1] + 1$.

Proof. From Theorem 8 we have that $R(B_{k,m}, K_n) = (k+m-1)(n-1) + 1$. Note that $R(B_{k,m}, K_2) = k+m$, so that

$$\begin{aligned} R(B_{k,m}, K_2, K_n) &= R(B_{k,m}, K_n) \\ &= [R(B_{k,m}, K_2) - 1](n-1) + 1. \end{aligned}$$

Hence we may apply Lemma 9 to get $R(B_{k,m}, K_{n_1}, \dots, K_{n_\ell}) = (k+m-1)[R(K_{n_1}, \dots, K_{n_\ell}) - 1] + 1$. \square

The authors are unsure whether a similar result can be proved for multiple bistars; we leave this as an open problem.

4 Bipartite Ramsey Numbers

Let G_1 and G_2 be bipartite graphs. Then $BR(G_1, G_2)$ is the least integer N so that any 2-coloring of the edges of $K_{N,N}$ contains either a red G_1 or a blue G_2 . In [5], Hattingh and Joubert determine the bipartite Ramsey number for certain bistars:

Theorem 11. *Let $k, n \geq 2$. Then $BR(B_{k,k}, B_{n,n}) = k + n - 1$.*

We generalize this result slightly.

Theorem 12. *Let $k \geq m \geq 2$, $n \geq \ell \geq 2$. Then $BR(B_{k,m}, B_{n,\ell}) = k + n - 1$.*

Proof. The upper bound follows immediately from Theorem 11. The lower bound construction given in Theorem 1 of Hattingh-Joubert for $BR(B_{s,s}, B_{t,t})$ does not work for us. We need this construction: Let L and R be the partite sets, and let $N = k + n - 2 = (k - 1) + (n - 1)$. Let $L = \{v_0, v_1, \dots, v_{N-1}\}$ and $R = \{w_0, w_1, \dots, w_{N-1}\}$. Color $v_i w_j$ *red* if $(i - j) \bmod N \in \{0, 1, \dots, k - 2\}$, and *blue* if $(i - j) \bmod N \in \{k - 1, \dots, N - 1\}$. Then the red subgraph is $(k - 1)$ -regular, hence no red $B_{k,m}$, and the blue subgraph is $(n - 1)$ -regular, hence no blue $B_{n,\ell}$. \square

Corollary 13. *Let T_m (resp., T_n) be a tree of diameter at most 3 with maximum degree m (resp., n). Then $BR(T_m, T_n) = m + n - 1$.*

Hattingh and Joubert also prove the following k -color upper bound.

Theorem 14. *For $k \geq 2$ and $m \geq 3$, we have*

$$BR_k(B_{m,m}) = BR(B_{m,m}, \dots, B_{m,m}) \leq \left\lceil k(m - 1) + \sqrt{(m - 1)^2(k^2 - k) - k(2m - 4)} \right\rceil$$

Hence $BR_k(B_{m,n}) = O(k)$. We provide a lower bound to get the following result.

Theorem 15. *Fix $m \geq 3$. Then $BR_k(B_{m,m}) = \Theta(k)$.*

Proof. We show that $BR_k(B_{m,m}) > k \cdot (m - 1)$. Let $N = k \cdot (m - 1)$, and consider a k -coloring of the edges of $K_{N,N}$ in colors c_0, \dots, c_{k-1} . Let the partite sets be $L = \{v_0, \dots, v_{N-1}\}$ and $R = \{w_0, \dots, w_{N-1}\}$. Color edge $v_i w_j$ with color c_ℓ if and only if $\ell \equiv (i - j) \bmod k$. Then the c_ℓ -subgraph is $(m - 1)$ -regular, hence there can be no monochromatic $B_{m,m}$. \square

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